“Proof” That $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$

A gloriously grubby piece of maths shown me by Andy Mayfield.

Maclaurin expansion of $\sin x$ is

$$\sin x \equiv x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Consider the expression $\dfrac{\sin x}{x}$,

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots .$$

By considering the graph of $y = \dfrac{\sin x}{x}$ we can see that the solutions of $0 = \dfrac{\sin x}{x}$ are $\pm n\pi$ for $n \in \mathbf{Z}^+$. So

$$\frac{\sin x}{x} \equiv 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \equiv \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\ldots .$$

Multiplying in pairs gives

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \equiv \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\ldots .$$

Considering the x^2 terms

$$-\frac{1}{3!} = -\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right),$$

and finally

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$